20 Periodic solutions. A first encounter with chaos

I already showed that discrete dynamical systems or maps can have periodic orbits. Here I will discuss them at some length.

Definition 1. *Consider a discrete dynamical system $x \mapsto f(x)$, $x \in U \subseteq \mathbf{R}$. A point $\hat{x}$ is called k-periodic or a periodic point with period k if $f^k(\hat{x}) = \hat{x}$.*

Recall that

$$f^k = \underbrace{f \circ \ldots \circ f}_{k \text{ times}}.$$

Therefore I have that k-periodic point is a fixed point of the k-th iteration of f and 1-periodic point is simply a fixed point of the map f. The orbit that starts at $\hat{x}$ and consists of exactly k points is called a periodic orbits. Note that *every* point of this orbit is k-periodic.

I am interested in *stable* periodic orbits. This is equivalent to studying the stability of a fixed point of the k-th iterate of f. The only thing is to check that the stability of the periodic orbit does not depend on the choice of a particular point.

To wit, let $\{x_1, x_2, \ldots, x_k\}$ be a periodic orbit. Consider,

$$\mu_i = \frac{\mathrm{d}f^k}{\mathrm{d}x}(x_i), \quad i = 1, \ldots, k.$$

Since $x_i = f(x_{i-1})$ for all $1 < i < k$ and $x_1 = f(x_k)$, I have

$$\mu_i = \frac{\mathrm{d}f\big(f^{k-1}(x_{i-1})\big)}{\mathrm{d}x} = f'(x_{i-1})\frac{\mathrm{d}f^{k-1}}{dx}(x_i) = \ldots = f'(x_{i-1})f'(x_{i-2})\ldots f'(x_i),$$

and does not depend on i. Therefore the stability condition for a periodic orbit takes the form

$$|f'(x_1)\ldots f'(x_k)| < 1.$$

In general, if a discrete map $x \mapsto f(x)$ is non-monotone then it may have quite intricate structure of periodic and non-periodic points. To formulate one of the most famous results, consider an ordering of all natural numbers:

$$\begin{aligned}
3 \succ 5 \succ 7 \succ \ldots \succ &\ \text{(all odd numbers except for 1)}\\
\succ 2\cdot 3 \succ 2 \cdot 5 \succ &\ \text{(all odd numbers except for 1 multiplied by 2)}\\
\succ 2^2\cdot 3 \succ 2^2 \cdot 5 \succ &\ \text{(all odd numbers except for 1 multiplied by } 2^2)\\
\succ \ldots \succ &\\
\succ 2^3 \succ 2^2 \succ 2 \succ 1.&
\end{aligned}$$

Theorem 2 (Sharkovsky[1])**.** *Consider continuous map $x \mapsto f(x)$ of the interval U into itself and assume that it has a k-periodic point. Then f has m-periodic points for all m such that $k \succ m$. In particular, if f has a 3-periodic point, it has orbits of any period.*

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[1]Sharkovsky, A. N. (1965). On cycles and the structure of a continuous mapping. Ukranian mathematical journal, 17(3), 104-111.

Sharkovsky's theorem was rediscovered in a famous paper by Li and Yorke, in which the term "chaos" was used for the first time to indicate that the behavior of the orbits is very irregular. The existence of periodic orbits of all possible periods is clearly an indication of a complicated behavior, however more was shown in the paper:

Theorem 3 (Li and Yorke[2]). *Consider continuous map $x \mapsto f(x)$ of the interval U into itself and assume that it has a 3-periodic point. Then there is an uncountable subset S of U such that every orbit starting in S is aperiodic and unstable.*

Example 4. Consider again the logistic family $x \mapsto rx(1-x) =: f_r(x)$ which maps $[0,1] \to [0,1]$ if $0 \leq r \leq 4$. I know that it admits periodic solutions of period 2 and 4. Let me show that there is $r \in [0,4]$ for which f_r has 3-periodic orbit. I need to find x_1, x_2, x_3 such that

$$x_2 = f_r(x_1), \quad x_3 = f_r(x_2), \quad x_1 = f_r(x_3),$$

and hence each of these points is a fixed point of f_r^3. The value of r when such fixed points can appear corresponds to the case when $\frac{\mathrm{d}f_r^3}{\mathrm{d}x}(x) = 1$ (see the figure). Therefore, I have a system of two equations with two unknowns

$$f_r^3(x) = x, \quad \frac{\mathrm{d}f_r^3}{\mathrm{d}x}(x) = 1,$$

which can be solved numerically[3] yielding

$$r \approx 3.8284, \quad x_1 \approx 0.1599, \quad x_2 \approx 0.5144, \quad x_3 \approx 0.9563.$$

You can also see this orbit of period three in the figure, where the cobweb diagram is shown (right panel).

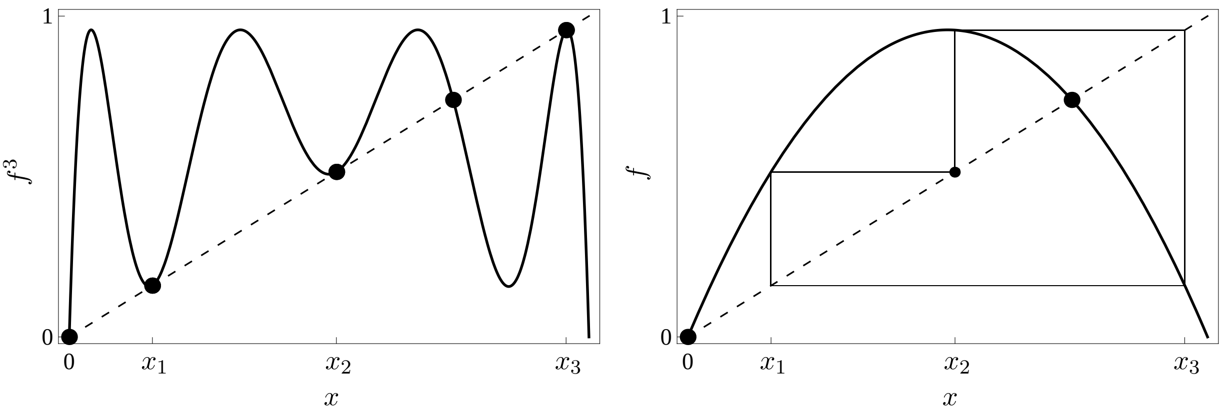

Figure 1: 3-periodic point in the logistic map $x \mapsto rx(1-x)$ for $r = 1 + \sqrt{8} \approx 3.83$

Example 5. Consider Ricker's map

$$x \mapsto rxe^{-x}, \quad r > 0,$$

[2]Li, T. Y., & Yorke, J. A. (1975). Period three implies chaos. American mathematical monthly, 985–992.

[3]It turns out that the exact value of r is $1 + \sqrt{8}$, but to prove that this is exactly the parameter value of 3-periodic point is not a simple problem

which maps $\mathbf{R}_+$ to $\mathbf{R}_+$. There is always fixed point $\hat{x}_0 = 0$, if $r > 1$ another fixed point appears, $\hat{x}_1 = \log r$. The multiplier of $\hat{x}_1$ is $\mu_1 = 1 - \log r$, therefore $\hat{x}_1$ is a sink if $1 < r < e^2$ and a source if $r > e^2$, when $r = e^2 =: r_1$ I have $\mu_1 = -1$ and I observe the flip bifurcation, which, as can be shown, is accompanied by appearance of a stable 2-periodic point. It can be checked numerically that for $r_2 \approx 12.51$ 2-periodic point loses stability via the flip bifurcation with appearance of a stable 4-periodic point. Next, for $r_4 \approx 14.24$ a stable 8-periodic point appears, and for $r_8 \approx 14.65$ a stable 16-periodic point is born via the same bifurcation (see the figure).

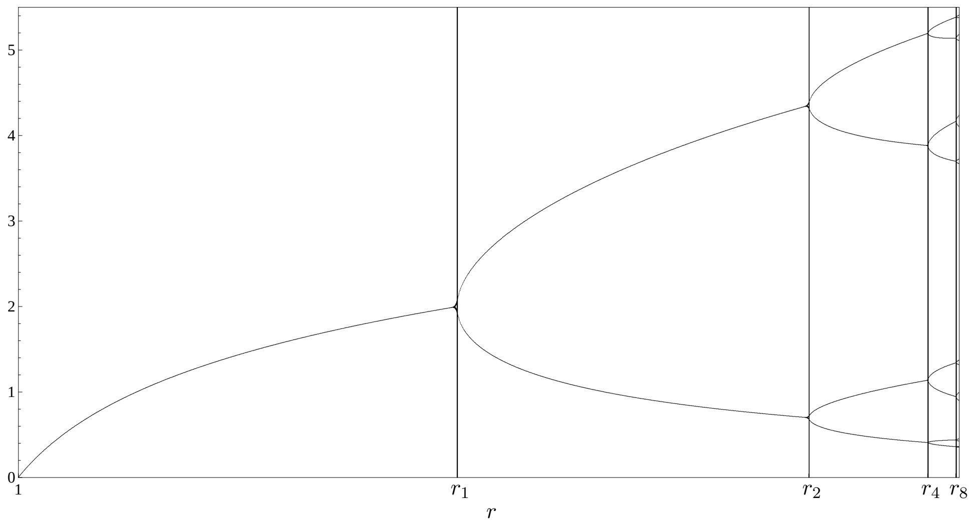

Figure 2: Period doubling or flip bifurcations in Ricker's map $x \mapsto rxe^{-x}$

It is natural to assume that there is an infinite sequence of the bifurcation parameter values r_{2^k}, $k = 0, 1, 2, \ldots$. It can be shown, actually, that

$$\frac{r_{2^k} - r_{2^{k-1}}}{r_{2^{k+1}} - r_{2^k}}$$

converges to a constant μ_F, called *Feigenbaum's constant*, which can be found as ≈ 4.6692. Moreover, the same constant appears in different maps and, hence, universal. Therefore, the sequence of flip, or period doubling, bifurcations, appears over and over again in discrete dynamical systems, and was called the period doubling rout to chaos (keep in mind that I did not even try to define what *chaos* means). In particular, it should be clear that the sequence of r_{2^k} converges quite fast to a limiting value r_∞ (for the logistic equation $r_\infty \approx 3.5699$).

This period doubling rout to chaos can be visualized as a bifurcation diagram, which is shown in the following figure for the logistic equation.

For comparison I also present full bifurcation diagram for Ricker's map

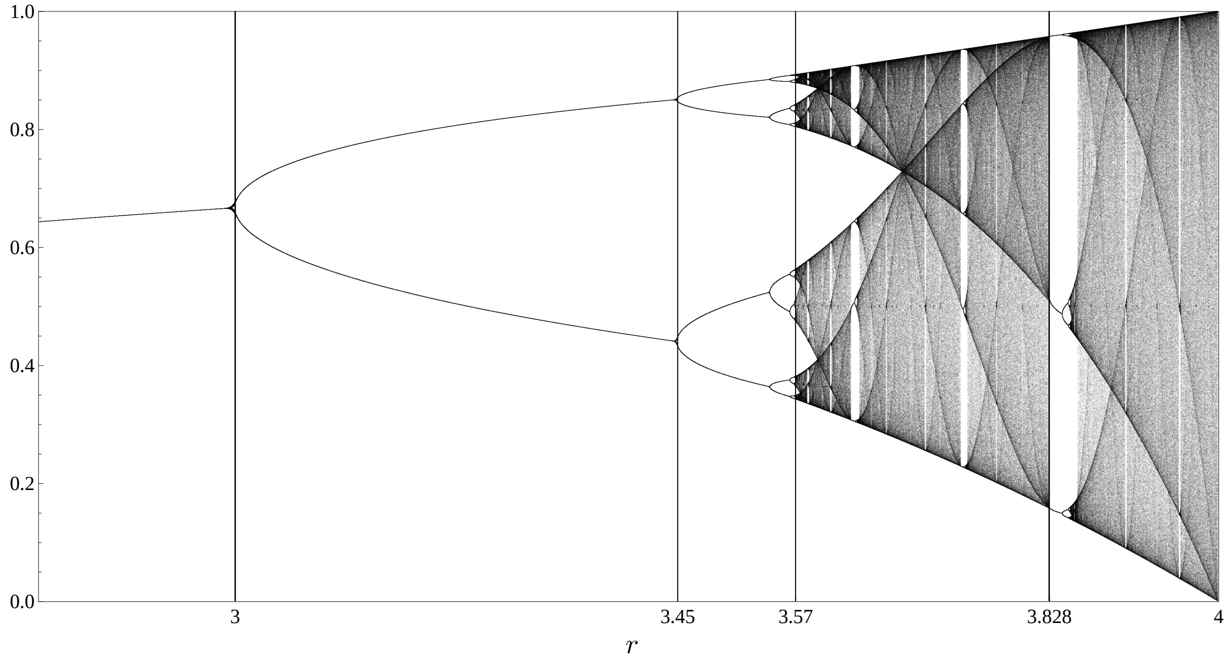

Figure 3: Bifurcation diagram for the logistic map $x \mapsto rx(1-x)$

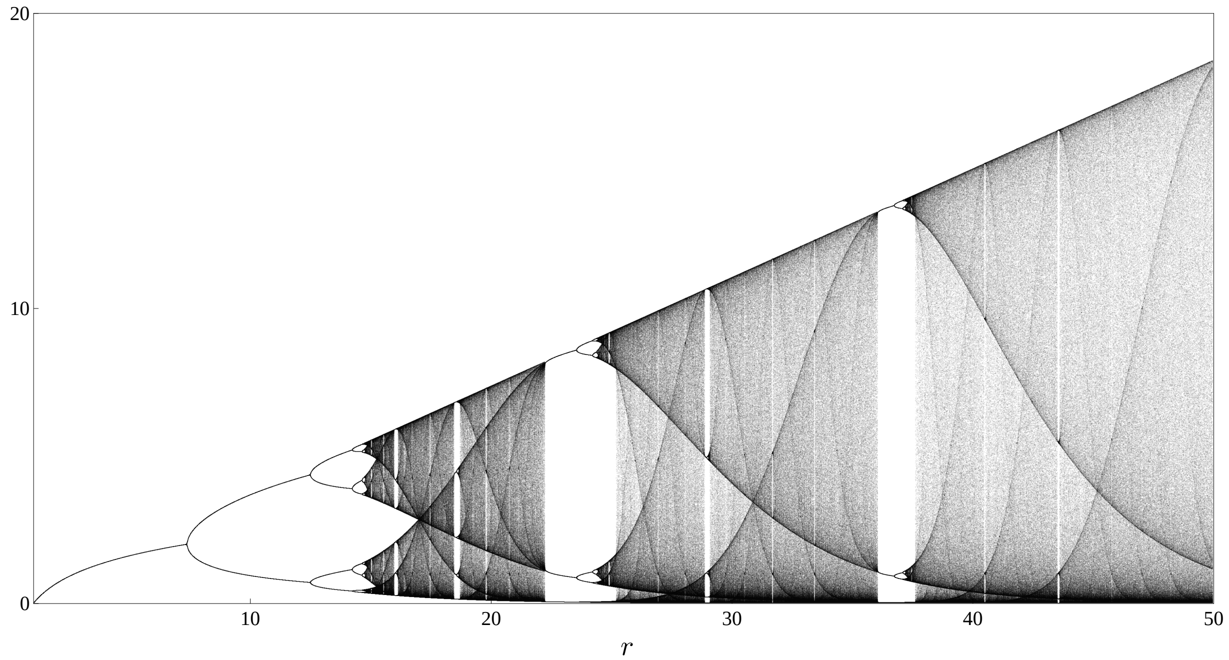

Figure 4: Bifurcation diagram for Ricker's map $x \mapsto rxe^{-x}$